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Gasdynamics of a centrifugal machine

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We consider axisymmetric steady centrifugally driven thermal convection in a compressible fluid in a rapidly rotating circular cylinder. The Boussinesq approximation is not used, because it is not valid for the case of practical interest. We clarify the importance of the effect of the flow-induced volume change of a fluid particle, and propose a widely applicable method of solution.

1. Introduction

The problem of centrifugally driven thermal convection in a compressible fluid in a circular cylinder is important in relation to a centrifugal machine used for the enrichment of uranium. Barcilon & Pedlosky (1967) and Homsy & Hudson (1969, 1971) studied this problem within the Boussinesq approximation. They clarified the point that the effect of thermal conduction is more predominant than the effect of thermal convection or vice versa according as $E^{-\frac{1}{2}}P_r(\tilde{\rho}_0-\tilde{\rho}_1)/\tilde{\rho}_0$ is smaller or larger than unity, where $E=\tilde{\mu}_0/\tilde{\rho}_0\,\Omega H^2$ is the Ekman number, $P_r=\tilde{\mu}C_p/\tilde{\kappa}$ the Prandtl number, $\tilde{\rho}$ the density, $\tilde{\mu}$ the viscosity, $\tilde{\kappa}$ the thermal conductivity, C_p the specific heat at constant pressure, Ω the angular velocity, H the height of the cylinder, and the suffixes 1 and 0 and tildes refer to typical points on the top and at the middle of the cylinder and to the original physical (dimensional) quantities, respectively. They also gave detailed discussions of the convection-dominated cases with $H\Omega^2/g=O(1)$ (Barcilon & Pedlosky 1967; Homsy & Hudson 1971) and with $H\Omega^2/g \gg 1$ (Homsy & Hudson 1969), where g is the gravitational acceleration.

These studies are valuable for gaining a qualitative understanding of the flow field. However, is the Boussinesq approximation quantitatively valid from a practical viewpoint? Let us consider a centrifugal machine of radius 30 cm rotating at 10 000–20 000 r.p.m. at room temperature, the working fluid being uranium fluoride, UF_6. If the fluid temperature is uniform and the fluid rotates rigidly with the container, the pressure or density scale height in the radial direction is 1–4 cm, which is smaller than the radius of the machine. This situation does not change, provided that the centrifugally driven thermal convection can be treated as a perturbation, except in regions too close to the axis. In such a case, the Boussinesq approximation is not valid from a quantitative viewpoint (Spiegel & Veronis 1960).

Our task in this paper is to study the effect of the non-Boussinesq compressibility of the working fluid on the centrifugally driven thermal convection. Let us consider a circular cylinder rotating around a vertical axis. The top temperature of the cylinder is higher than the bottom temperature, and the side-wall temperature changes linearly with height. This linear distribution of temperature may be achieved if the thermal conductivity of the cylinder is sufficiently high in comparison with that of the fluid. The temperature difference between the top and the bottom is so small that the thermal Rossby number $\delta = (\tilde{T}_1 - \tilde{T}_0)/\tilde{T}_0$ (where $\tilde{T}$ is the temperature) may be treated as an infinitesimal. Our problem is to study the axisymmetric flow of compressible fluid in this cylinder. We assume that the angular velocity is so large that the radial scale height is smaller than the radius, except near the axis. The vertical scale height, on the contrary, is much larger than the height of the cylinder. Thus, we discard the Boussinesq approximation, and use the original equations of state and of continuity. We assume that the viscosity and the thermal conductivity are small and depend only on the temperature, and that the Prandtl number and the ratio of the radius to the height of the cylinder are of order unity.

Before beginning the detailed mathematical discussion, let us have a quick glance at aspects characteristic of non-Boussinesq compressibility. Because the radial scale height is smaller than the radius of the cylinder, a fluid particle appreciably swells or shrinks when it moves radially. The interaction between the fluid particle and its environment via the work due to this volume change gives a new kind of coupling between the velocity and the temperature. Thus, the temperature changes simultaneously with the velocity in the boundary layers. Another interesting aspect is that the thickness of the horizontal boundary layer is proportional to the reciprocal of the square root of the density. We can ascribe this to the fact that the thickness of the horizontal layer is proportional to the square root of the kinematic viscosity, and that we have assumed the viscosity to depend only on the temperature.

In § 2 we describe the linearized basic equations for a small perturbation to a basic state of rigid-body rotation with uniform temperature. We define the basic parameters and express our restriction to the case $E^{\frac{1}{2}} \gg G$, where G is the ratio of the height of the cylinder to the vertical scale height. We discuss the boundary layers in § 3, and the temperature field of the main bulk of the inner flow in § 4. In § 5, we give numerical results and a discussion.

2. Basic equations

Because of our assumption that the thermal Rossby number is infinitesimally small, we can treat the convection as a small perturbation to a basic state of rigid-body rotation with uniform temperature. We can determine the pressure $\tilde{p}$ and the density $\tilde{\rho}$ of the basic state by balancing the forces in a rotating frame of reference:

$$\left.\begin{aligned} \tilde{p}_B = \tilde{p}_0\epsilon_B, \quad \tilde{\rho}_B = \tilde{\rho}_0\epsilon_B, \quad \tilde{p}_0 = \tilde{\rho}_0 R\tilde{T}_0, \\ \epsilon_B = \exp\{\tilde{r}^2\Omega^2/2R\tilde{T}_0 - g(\tilde{z} - \tfrac{1}{2}H)/R\tilde{T}_0\}, \end{aligned}\right\} \tag{2.1}$$

where $(\tilde{r}, \theta, \tilde{z})$ is a rotating system of cylindrical co-ordinates with angular velocity Ω, $\tilde{T}_0$ the temperature, R the gas constant and the suffixes 0 and B refer to the centre of the midplane and to the basic state, respectively. Because $R\tilde{T}_0/\Omega^2\tilde{r}$ is the radial scale height, the basic pressure and the density change appreciably with distance from the axis because of our assumption that the radial scale height is smaller than the cylinder radius. Thus, we cannot use Boussinesq approximation (Spiegel & Veronis 1960).

By using the non-dimensional quantities

$$\left.\begin{aligned}(r,z) = (\tilde{r}/H, \tilde{z}/H), \quad (u,v,w) = (\Omega H/\delta R\tilde{T}_0)(\tilde{q}_r, \tilde{q}_\theta, \tilde{q}_z),\\ T = (\tilde{T}-\tilde{T}_0)/\delta\tilde{T}_0, \quad p = (\tilde{p}-\tilde{p}_B)/\delta\tilde{p}_B, \quad \rho = (\tilde{\rho}-\tilde{\rho}_B)/\delta\tilde{\rho}_B,\end{aligned}\right\} \tag{2.2}$$

where $(\tilde{q}_r, \tilde{q}_\theta, \tilde{q}_z)$ is the velocity in the rotating frame, and by neglecting terms of higher order in δ, we obtain linearized basic equations for a perturbation to the above basic state:

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z}+G_0 ru - Gw = 0, \tag{2.3}$$

$$-2v+G_0 rT+\frac{\partial p}{\partial r} = \frac{E}{\epsilon_B}\left\{Lu+\frac{1}{3}\frac{\partial}{\partial r}\nabla.\mathbf{V}\right\}, \tag{2.4}$$

$$2u = (E/\epsilon_B)Lv, \tag{2.5}$$

$$-GT+\frac{\partial p}{\partial z} = \frac{E}{\epsilon_B}\left\{\Delta w+\frac{1}{3}\frac{\partial}{\partial z}\nabla.\mathbf{V}\right\}, \tag{2.6}$$

$$\left(\frac{\Gamma-1}{\Gamma}\right)P_r\left\{-ru+\frac{G}{G_0}w\right\} = \frac{E}{\epsilon_B}\Delta T, \tag{2.7}$$

$$p = \rho+T, \tag{2.8}$$

where

$$\left.\begin{aligned}\nabla.\mathbf{V} &= \frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z},\\ \Delta &= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}, \quad L = \Delta-\frac{1}{r^2},\\ E &= \frac{\tilde{\mu}_0}{\tilde{\rho}_0\Omega H^2}, \quad G_0 = H^2\Omega^2/R\tilde{T}_0, \quad G = gH/R\tilde{T}_0\end{aligned}\right\} \tag{2.9}$$

and Γ is the ratio of the specific heats.

We can obtain boundary conditions by expressing the fact that the top and the bottom temperatures are uniform, the side-wall temperature changes linearly with height and that the fluid velocity coincides with that of the cylinder on the cylinder surface:

$$u = v = w = 0, \quad T = (-1)^{j+1} \quad \text{on} \quad z = j \quad (j = 0,1), \quad (0 \leqslant r \leqslant r_0), \tag{2.10}$$

$$u = v = w = 0, \quad T = 2z-1 \quad \text{on} \quad 0 \leqslant z \leqslant 1 \quad (r = r_0). \tag{2.11}$$

To the same order of approximation, the left-hand side of (2.7) vanishes in the Boussinesq approximation. This term reproduces the work done by a fluid particle via its volume change, and is crucial for the flow of non-Boussinesq compressible fluid. We can estimate the main parameters E, G_0 and G corresponding to the size and the operating conditions of the centrifugal machine described in § 1 to be of order 10^{-7}, 10 and 10^{-4}, respectively. According to this estimate, and for the sake of mathematical simplicity, we restrict ourselves to the case with $E^{\frac{1}{2}} \gg G$, and with $G_0 \sim 1$.

3. Boundary-layer analysis

Because we consider a fluid with small viscosity and thermal conductivity, which is represented by the smallness of E in the basic equations, we may divide the flow field into the horizontal and the vertical boundary layers and the main inner flow.

Main inner flow

Because of the centrifugal buoyancy, light fluid near the high temperature top floats towards the axis, and heavy fluid near the low temperature bottom sinks towards the side wall. Owing to the continuity of the flow, this radial flow in the horizontal boundary layer drives the meridional current in the main inner flow. To take into account this driving mechanism, we assume that the axial velocity and the temperature in the inner flow are of the same order of magnitude as those in the horizontal boundary layer, which are of order $E^{\frac{1}{2}}$ and unity, respectively, as will be shown in equation (3.11). Thus, the scaling law of the inner flow is

$$\left.\begin{aligned} u_i &= E^{-1}u, \quad v_i = v, \quad w_i = E^{-\frac{1}{2}}w, \\ T_i &= T, \quad p_i = p, \quad \rho_i = \rho, \end{aligned}\right\} \tag{3.1}$$

where the suffix i denotes order-unity quantities in the inner flow. The lowest order equations, with respect to E and G, for the inner flow are

$$\partial w_i/\partial z = 0, \tag{3.2}$$

$$-2v_i + G_0 r T_i = -\partial p_i/\partial r, \tag{3.3}$$

$$2\epsilon_B u_i = L v_i, \tag{3.4}$$

$$\partial p_i/\partial z = 0, \tag{3.5}$$

$$-ru_i = [\Gamma/(\Gamma-1)P_r\epsilon_B]\,\Delta T_i, \tag{3.6}$$

$$p_i = \rho_i + T_i. \tag{3.7}$$

Because the side-wall temperature is anti-symmetric with respect to $z = \frac{1}{2}$, we can restrict ourselves to inner flow temperatures with the same symmetry character. Then, p_i vanishes by (3.3) and (3.5), and we obtain

$$v_i = \tfrac{1}{2}G_0 r T_i. \tag{3.8}$$

We can obtain u_i, v_i and ρ_i from (3.4), (3.7) and (3.8) once we have found T_i. A single equation for T_i is obtained by an elimination procedure:

$$(1+hr^2)\left(\frac{\partial^2 T_i}{\partial r^2} + \frac{\partial^2 T_i}{\partial z^2}\right) + \frac{1+3hr^2}{r}\frac{\partial T_i}{\partial r} = 0, \tag{3.9}$$

where
$$h = (\Gamma-1)P_r G_0/4\Gamma. \tag{3.10}$$

To determine the boundary conditions on (3.9), and to obtain w_i from (3.2) and the compatibility condition on the horizontal boundary, let us proceed to the boundary-layer analysis.

Horizontal boundary layer

Through the strong coupling between the velocity and the temperature via the left-hand side of (2.7), the temperature changes simultaneously with the velocity in the boundary layer. The scaling law of the horizontal boundary layer for the case with $E^{\frac{1}{2}} \gg G$ is as follows:

$$\left.\begin{aligned} &\hat{u} = u, \quad \hat{v} = v, \quad \hat{w} = E^{-\frac{1}{2}}w, \quad \hat{T} = T, \quad \hat{\rho} = \rho, \\ &\hat{p} = E^{-1}p, \quad \zeta_j = (-1)^j E^{-\frac{1}{2}}(z-j) \quad (j = 0, 1), \end{aligned}\right\} \tag{3.11}$$

where the carets denote order-unity quantities in the horizontal boundary layer. The boundary-layer equations are

$$\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + G_0 r \hat{u} + (-1)^j \frac{\partial \hat{w}}{\partial \zeta_j} = 0, \tag{3.12}$$

$$-2\hat{v} + G_0 r \hat{T} - \frac{1}{\epsilon_B} \frac{\partial^2 \hat{u}}{\partial \zeta_j^2} = 0, \tag{3.13}$$

$$2\epsilon_B \hat{u} - \partial^2 \hat{v}/\partial \zeta_j^2 = 0, \tag{3.14}$$

$$(-1)^j \frac{\partial \hat{p}}{\partial \zeta_j} = \frac{1}{\epsilon_B} \left\{ \frac{4}{3} \frac{\partial^2 \hat{w}}{\partial \zeta_j^2} + \frac{(-1)^j}{3r} \frac{\partial}{\partial r} \left(r \frac{\partial \hat{u}}{\partial \zeta_j} \right) \right\}, \tag{3.15}$$

$$r\hat{u} + \frac{\Gamma}{(\Gamma - 1) P_r \epsilon_B} \frac{\partial^2 \hat{T}}{\partial \zeta_j^2} = 0, \tag{3.16}$$

$$\hat{\rho} + \hat{T} = 0, \tag{3.17}$$

where we retain only the lowest order terms with respect to E and G.

The boundary conditions are

$$\hat{u} = v_i + \hat{v} = w_i + \hat{w} = 0, \quad T_i + \hat{T} = (-1)^{j+1} \quad \text{at} \quad \zeta_j = 0 \quad (j = 0, 1), \tag{3.18}$$

$$\hat{f} \to 0 \quad \text{as} \quad \zeta_j \to \infty, \tag{3.19}$$

for any function f. Eliminating $\hat{u}$ from (3.14) and (3.16), and remembering the boundary conditions at $\zeta_j \to \infty$, we obtain

$$\hat{T} = -(\Gamma - 1) P_r r \hat{v}/2\Gamma. \tag{3.20}$$

Elimination of $\hat{T}$ and $\hat{v}$ from (3.13), (3.14) and (3.18) gives us a single ordinary differential equation for $\hat{u}$:

$$\partial^4 \hat{u}/\partial \zeta_j^4 + 4\sigma^4 \hat{u} = 0, \tag{3.21}$$

where

$$\sigma = \{\epsilon_B^2 (1 + hr^2)\}^{\frac{1}{4}}. \tag{3.22}$$

The solution of (3.21) subject to (3.18) and (3.19) gives us the following:

$$\hat{u} = (-1)^j G_0 r e^{-\sigma\zeta_j} \sin \sigma\zeta_j / 2(1 + hr^2)^{\frac{1}{2}}, \tag{3.23}$$

$$\hat{T} = (-1)^{j+1} hr^2 e^{-\sigma\zeta_j} \cos \sigma\zeta_j / (1 + hr^2), \tag{3.24}$$

$$\hat{v} = (-1)^j G_0 r e^{-\sigma\zeta_j} \cos \sigma\zeta_j / 2(1 + hr^2), \tag{3.25}$$

where use has been made of (3.8), (3.13) and (3.20). From the above expressions and (3.18), we obtain

$$w_i = -\frac{G_0}{8\epsilon_B^{\frac{1}{2}}(1+hr^2)^{\frac{3}{4}}}\left(1+G_0 r^2+\frac{3}{1+hr^2}\right), \tag{3.26}$$

$$T_i = (-1)^{j+1}/(1+hr^2) \quad \text{at} \quad z = j, \tag{3.27}$$

where we have used (3.2) and (3.12) to obtain w_i.

As we see from (3.13), the Coriolis forces, the centrifugal buoyancy and the viscous forces play roles in the horizontal boundary layer. We can consider the horizontal layer as a hybrid of the buoyancy and the Ekman layer.

Another interesting aspect of the layer is that the thickness parameter σ depends on r as we see in (3.22). Thus, the thickness of the layer is proportional to the reciprocal of the square root of the basic density; the layer is thickest at the axis and becomes thinner as the radial distance increases. As we explained in the introduction, this is ascribed to the fact that the thickness of the horizontal layer is proportional to the square root of the kinematic viscosity, and that we have assumed the viscosity to depend only on the temperature.

Side-wall boundary layer

Because the side-wall temperature is anti-symmetric with respect to $z = \frac{1}{2}$, the Stewartson $E^{\frac{1}{4}}$-layer does not appear in our case (Stewartson 1957). The scaling law of the side-wall boundary layer is as follows:

$$\overline{u} = E^{-\frac{1}{3}}u, \quad \overline{v} = v, \quad \overline{w} = w, \quad \overline{T} = T, \quad \overline{\rho} = \rho, \quad \overline{p} = E^{-\frac{1}{3}}p, \quad \eta = E^{-\frac{1}{3}}(r_0 - r), \tag{3.28}$$

where overbars denote order-unity quantities in the side-wall boundary layer. The boundary-layer equations are

$$\partial\overline{w}/\partial z - \partial\overline{u}/\partial\eta = 0, \tag{3.29}$$

$$-2\overline{v} + G_0 r_0 \overline{T} - \partial\overline{p}/\partial\eta = 0, \tag{3.30}$$

$$2\epsilon_B \overline{u} - \partial^2\overline{v}/\partial\eta^2 = 0, \tag{3.31}$$

$$\frac{\partial\overline{p}}{\partial z} - \frac{1}{\epsilon_B}\frac{\partial^2\overline{w}}{\partial\eta^2} = 0, \tag{3.32}$$

$$r_0\overline{u} + \frac{\Gamma}{(\Gamma-1)P_r\epsilon_B}\frac{\partial^2\overline{T}}{\partial\eta^2} = 0, \tag{3.33}$$

$$\overline{\rho} + \overline{T} = 0, \tag{3.34}$$

and the boundary conditions are

$$T_i + \overline{T} = 2z - 1, \quad v_i + \overline{v} = 0, \quad \overline{u} = \overline{w} = 0 \quad \text{at} \quad \eta = 0, \tag{3.35}$$

for all f,

$$\overline{f} \to 0 \quad \text{as} \quad \eta \to \infty \tag{3.36}$$

$$\overline{w} = 0 \quad \text{at} \quad z = j \quad (j = 0, 1), \tag{3.37}$$

where we again retain the lowest order terms with respect to E and G.

Elimination of $\overline{u}$ from (3.31) and (3.33) subject to (3.36) gives us

$$\overline{T} = -(\Gamma-1)P_r r_0 \overline{v}/2\Gamma. \tag{3.38}$$

From (3.8), (3.38) and (3.35), we obtain

$$T_i = (2z-1)/(1+hr_0^2) \quad \text{at} \quad r = r_0, \tag{3.39}$$

$$\overline{T} = hr_0^2(2z-1)/(1+hr_0^2) \quad \text{at} \quad \eta = 0. \tag{3.40}$$

Let us eliminate $\overline{p}$, $\overline{u}$ and $\overline{w}$ from (3.29), (3.30), (3.31), (3.32) and (3.38) to obtain

$$\frac{\partial^6 \overline{T}}{\partial \eta^6} + 4\epsilon_B^2(1+hr_0^2)\frac{\partial^2 \overline{T}}{\partial z^2} = 0. \tag{3.41}$$

It is interesting that we obtain a partial differential equation instead of an ordinary differential equation. Because $\overline{w}$ is symmetric with respect to $z = \frac{1}{2}$, we see that $\overline{u}$, $\overline{v}$ and $\overline{T}$ are anti-symmetric, by inspection of the boundary-layer equations. Then, the procedure for determining the boundary-layer quantities is similar to that of Hunter (1967). The function $\overline{T}$ is expanded in a cosine series as

$$\overline{T} = \sum_{n=0}^{\infty} f_n(\eta)\cos(2n+1)\pi z. \tag{3.42}$$

Substitution of (3.42) into (3.41) gives us

$$d^6 f_n/d\eta^6 = 4\epsilon_B^2(1+hr_0^2)(2n+1)^2\pi^2 f_n. \tag{3.43}$$

The boundary conditions (3.35), (3.36) and (3.40) give us

$$f_n(0) = -\frac{8}{(2n+1)^2\pi^2}\frac{hr_0^2}{1+hr_0^2}, \quad f_n''(0) = f_n''''(0) = 0, \tag{3.44}$$

$$f_n \to 0 \quad \text{as} \quad \eta \to \infty. \tag{3.45}$$

The solution of (3.43) subject to (3.44) and (3.45) is as follows:

$$f_n(\eta) = -\frac{4}{(2n+1)^2\pi^2}\frac{hr_0^2}{1+hr_0^2}\left[e^{-\omega_n\eta} + e^{-\frac{1}{2}\omega_n\eta}\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}\omega_n\eta - \frac{\pi}{6}\right)\right], \tag{3.46}$$

where $$\omega_n = [2\epsilon_B(2n+1)\pi]^{\frac{1}{3}}(1+hr_0^2)^{\frac{1}{6}}.$$

In the Ekman extensions of the side-wall boundary layer, which are within a distance $O(E^{\frac{1}{2}})$ of the top and the bottom, we have to make a careful examination. Because the results of such an examination do not affect the above procedure, however, we omit the discussion of the Ekman extension. What we have to do now is to solve (3.9) subject to (3.27) and (3.39).

4. The solution for the main inner flow

Let us decompose T_i as

$$T_i = \frac{2z-1}{1+hr_0^2} + \sum_{n=1}^{\infty}\frac{\sinh(\lambda_n z) - \sinh\{\lambda_n(1-z)\}}{\sinh\lambda_n}T_n(r). \tag{4.1}$$

Substitution of (4.1) into (3.9), (3.27) and (3.39) gives us

$$\frac{d^2T_n}{dr^2}+\frac{1}{r}\frac{dT_n}{dr}+\lambda_n^2T_n = -\frac{2hr}{1+hr^2}\frac{dT_n}{dr}, \tag{4.2}$$

$$\sum_{n=1}^{\infty} T_n(r) = \frac{1}{1+hr^2}-\frac{1}{1+hr_0^2}, \tag{4.3}$$

$$T_n(r_0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{4.4}$$

Our problem is reduced to the solution of (4.2) subject to (4.3) and (4.4). By the Sturm–Liouville theorem, the solutions of (4.2) subject to (4.4) form an orthogonal set of functions $\{T_n(r)\}$. Thus, (4.3) can be satisfied term by term by the standard method for an orthogonal set of functions.

Because $\Gamma-1$ is small for the case of practical interest (about 0·07 for UF_6), we can take $h\,[=(\Gamma-1)P_rG_0/4\Gamma]$ as a small parameter and expand T_n and λ_n in terms of h:

$$T_n = hT_{n0}+h^2T_{n1}+\ldots, \quad \lambda_n = \lambda_{n0}+h\lambda_{n1}+\ldots. \tag{4.5}$$

Substitution of (4.5) into (4.2)–(4.4) gives us

$$\frac{d^2T_{n0}}{dr^2}+\frac{1}{r}\frac{dT_{n0}}{dr}+\lambda_{n0}^2T_{n0} = 0, \tag{4.6}$$

$$\frac{d^2T_{n1}}{dr^2}+\frac{1}{r}\frac{dT_{n1}}{dr}+\lambda_{n0}^2T_{n1} = -2\lambda_{n1}\lambda_{n0}T_{n0}-2r\frac{dT_{n0}}{dr}, \tag{4.7}$$

$$T_{n0}(r_0) = T_{n1}(r_0) = 0, \tag{4.8}$$

$$\sum_{n=1}^{\infty} T_{n0}(r) = r_0^2-r^2, \quad \sum_{n=1}^{\infty} T_{n1}(r) = r^4-r_0^4, \tag{4.9}$$

where we show the equations for the lowest two terms of the expansions. The solution of (4.6) subject to (4.8) is

$$T_{n0}(r) = a_{n0}J_0(\lambda_{n0}r), \tag{4.10}$$

where a_{n0} is a coefficient to be determined by (4.9), and

$$\lambda_{n0} = j_{0n}/r_0, \tag{4.11}$$

where j_{0n} is the nth positive zero of $J_0(x)$. Because $\{T_{n0}(r)\}$ is an orthogonal set of functions, as a special case of the above-mentioned orthogonality property, we can determine the a_{n0} as follows:

$$a_{n0} = 8r_0^2/j_{0n}^3J_1(j_{0n}). \tag{4.12}$$

The solution of (4.7) subject to (4.8) is

$$\begin{aligned} T_{n1} &= a_{n1}J_0(\lambda_{n0}r)+a_{n0}[\tfrac{1}{2}r^2J_2(\lambda_{n0}r)-\lambda_{n1}rJ_1(\lambda_{n0}r)] \\ &= (a_{n1}-\tfrac{1}{2}r^2a_{n0})J_0(\lambda_{n0}r). \end{aligned} \tag{4.13}$$

The condition (4.8) gives us

$$\lambda_{n1} = 1/\lambda_{n0}. \tag{4.14}$$

We can determine the a_{n1} from (4.9):

$$a_{n1} = -\frac{4r_0^4}{3j_{0n}^3J_1(j_{0n})}\left(23-\frac{94}{j_{0n}^2}\right)+\frac{32j_{0n}r_0^4}{J_1(j_{0n})}\sum_{\substack{m=1\\(m\neq n)}}^{\infty}\frac{1}{j_{0m}^2(j_{0n}^2-j_{0m}^2)}. \tag{4.15}$$

Performing the summation in (4.15), we have

$$a_{n1} = -\frac{4r_0^4}{j_{0n}^3 J_1(j_{0n})}\left(5-\frac{16}{j_{0n}^2}\right). \tag{4.16}$$

Inserting (4.12) and (4.16) into (4.13), we obtain

$$T_{n1} = -\frac{4r_0^4}{j_{0n}^3 J_1(j_{0n})}\left[5+\left(\frac{r}{r_0}\right)^2-\frac{16}{j_{0n}^2}\right] J_0\left(j_{0n}\frac{r}{r_0}\right). \tag{4.17}$$

Summarizing, the solution for T_i can be expressed as follows:

$$T_i = (2z-1)/(1+hr_0^2)+hr_0^2 T^{(1)}+(hr_0^2)^2 T^{(2)}+\dots, \tag{4.18}$$

where

$$T^{(1)} = \sum_{n=1}^{\infty} \frac{\sinh(\lambda_{n0}z)-\sinh[\lambda_{n0}(1-z)]}{\sinh\lambda_{n0}} \frac{8}{j_{0n}^3 J_1(j_{0n})} J_0\left(j_{0n}\frac{r}{r_0}\right), \tag{4.19}$$

$$\begin{aligned} T^{(2)} = &-\sum_{n=1}^{\infty} \frac{\sinh(\lambda_{n0}z)-\sinh[\lambda_{n0}(1-z)]}{\sinh\lambda_{n0}} \frac{4}{j_{0n}^3 J_1(j_{0n})}\left[5+\left(\frac{r}{r_0}\right)^2-\frac{16}{j_{0n}^2}\right] J_0\left(j_{0n}\frac{r}{r_0}\right) \\ &+\sum_{n=1}^{\infty}\frac{1}{\sinh\lambda_{n0}}\{z\cosh\lambda_{n0}z-(1-z)\cosh\lambda_{n0}(1-z) \\ &-\coth\lambda_{n0}[\sinh\lambda_{n0}z-\sinh\lambda_{n0}(1-z)]\}\frac{8}{r_0 j_{0n}^4 J_1(j_{0n})} J_0\left(j_{0n}\frac{r}{r_0}\right). \end{aligned} \tag{4.20}$$

5. Results and discussion

Because the convergence of the series in (4.19) and (4.20) is fast, as is expected owing to the factor j_{0n}^3 in the denominator, we can truncate them after the first twenty terms. We can see the validity of the truncation from the agreement of thus calculated values of $-T^{(1)}$ and $T^{(2)}$ with the exact values of $1-(r/r_0)^2$ and $1-(r/r_0)^4$, respectively, on $z = 0$. Figures 1 and 2 show $T^{(1)}$ and $T^{(2)}$ for $r_0 = 1$, 0·1 and 10.

As we see in (4.18), the present expansion parameter is

$$hr_0^2 = 0{\cdot}25(\Gamma-1)P_r G_0 r_0^2/\Gamma$$

rather than h. The parameter can become small if $(\Gamma-1)r_0^2$ is small for the case with order-unity G_0. Thus, our approximation procedure is expected to have a wide range of applicability for operating gases of small $\Gamma-1$ in a machine of small aspect ratio r_0. In effect, for the case of UF_6 in the machine with $r_0 = \frac{1}{3}$, G_0 is about 70 for hr_0^2 of order 0·1.

Finally, let us consider the meaning of our restrictions to the case with

$$GE^{-\frac{1}{2}} \ll 1 \quad \text{and} \quad \delta \ll 1.$$

We can omit the part of the temperature distribution symmetric with respect to $z = \frac{1}{2}$ because of the first restriction. If we relax this restriction, we must take into account the symmetric part also, and thus the effect of the $E^{\frac{1}{4}}$ side-wall layer as Hunter (1967) noted. The second restriction ensures the validity of our linearized treatment. If we abandon this restriction, we must take into account the nonlinear effect of the thermal convection as was done by Barcilon & Pedlosky (1967) and Homsy & Hudson (1969).

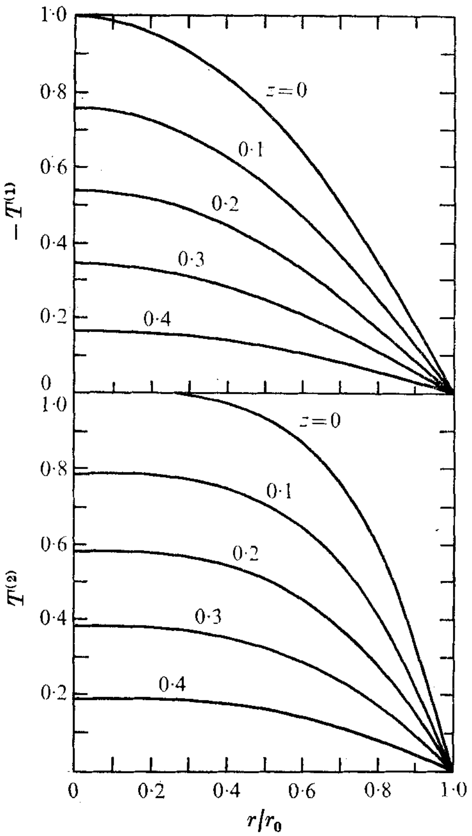

FIGURE 1. Distributions of the expansion coefficients of the temperature field $-T^{(1)}$ and $T^{(2)}$ in the inner flow. The aspect ratio r_0 is taken to be 1.

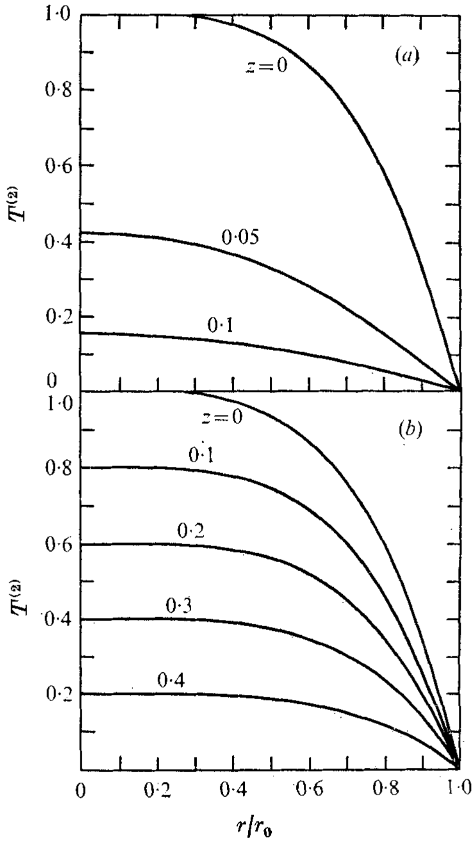

FIGURE 2. Distributions of $T^{(2)}$ for (a) $r_0 = 0{\cdot}1$ and (b) $r_0 = 10$.

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